Last time: Augmented matrices

Reduced Row Echelon Form ←

Matrix Ops.

Matrix Operations

Refresh: Matrix addition: Given A and B matrizes of the same size $m \times n$, their sum is computed entry-wise.

Ex: $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+3 & 1-3 \\ 0+2 & -1+1 \\ -1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 0 \\ -1 & 0 \end{bmatrix}$

Non Ex: $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ is UNDEFINED!

Def^n: Given constant (or scalar) c and matrix A, the scalar multiple of A by c is cA w/ entries the componentwise product (c by entry).

Ex: $-2 \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 4 & -6 \\ 8 & -14 \end{bmatrix}$

Def^n: Given matrices A and B of sizes $m \times k$ and $k \times n$ respectively, the matrix product $A \cdot B$ is computed by: $A \cdot B = [a_{i,j}]_{i,j} \cdot [b_{i,j}]_{i,j} = \left[\sum_{p=1}^{k} a_{i,p} b_{p,j} \right]_{i,j}$

Ex: Compute AB for $A = \begin{bmatrix} 3 & 0 & -1 \\ 5 & -5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$

Sol:

$$\underset{2\times 3}{\begin{bmatrix} \boxed{3 \quad 0 \quad -1} \\ 5 \quad -5 \quad 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}} = \begin{bmatrix} 3\cdot 1 + 0\cdot 1 + -1\cdot 0 & 3\cdot -1 + 0\cdot 1 + -1\cdot -2 \\ 5\cdot 1 + -5\cdot 1 + 0\cdot 0 & 5\cdot -1 + -5\cdot 1 + 0\cdot -2 \end{bmatrix}$$

$$= \underset{2\times 2}{\begin{bmatrix} 3 & -1 \\ 0 & -10 \end{bmatrix}} \qquad \blacksquare$$

NB: Size of product is determined by sizes of factors...

Ex: Product $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ by $\begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 3 \\ -2 & 2 & -1 \\ -3 & 0 & 0 \end{bmatrix}$.

Sol:

$$\underset{2\times 4}{\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix}} \underset{4\times 3}{\begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 3 \\ -2 & 2 & -1 \\ -3 & 0 & 0 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 6 & -3 & 3 \\ -11 & 4 & 0 \end{bmatrix}} \qquad \blacksquare$$

In general, an $m \times \boxed{k}$ matrix times a $\boxed{k} \times n$ matrix results in an $m \times n$ matrix.

Ex: Multiply $\overset{3\times 1}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$ by $\overset{1\times 4}{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}}$.

Sol:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad \blacksquare$$

Ex: Let $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ -1 & -3 \end{bmatrix}$.

First compute $A \cdot B$, then compute $B \cdot A$.

Sol:

$$\underset{2\times 2}{AB = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 3 & 0 \\ -1 & -3 \end{bmatrix}} = \begin{bmatrix} 5 & -3 \\ -6 & 0 \end{bmatrix}$$

different!

$$BA = \underset{2\times 2}{\begin{bmatrix} 3 & 0 \\ -1 & -3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}} = \begin{bmatrix} 6 & 3 \\ 4 & -1 \end{bmatrix}$$

This example demonstrates that matrix multiplication is <u>NOT</u> commutative (i.e. order matters!). ☐

<u>NB</u>: Suppose A is an $m \times n$ matrix and $\vec{x}$ is an $n \times 1$ matrix (i.e. column vector)

$A\vec{x}$ is an $m \times 1$ matrix. We can use this observation to build a third rep. of a linear system. Suppose our linear system has a rep via augmented matrices:

$$[A \mid \vec{b}] \text{ where } A \text{ is } m \times n \text{ and } \vec{b} \text{ is } m \times 1.$$

If we let $\vec{x}$ denote the vector of system variables, this augmented matrix also represents the equation $A\vec{x} = \vec{b}$.

Ex: Represent linear system $\begin{cases} x + y - z = 3 \\ x - y + z = 2 \\ x + y + z = 1 \end{cases}$ by a matrix equation (and by an augmented matrix.

Sol: The system has augmented matrix

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array} = [A \mid \vec{b}]$$

matrix of coefficients

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$ so the system

has matrix equation $A\vec{x} = \vec{b}$ i.e. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ☑

We'll think about linear systems in terms of matrix equations from now on :).

Homogeneous and Nonhomogeneous Systems

Def^n: A linear system $A\vec{x} = \vec{b}$ is <u>homogeneous</u> when $\vec{b} = \vec{0}$ (i.e. $\vec{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} =: \vec{0}$).

Ex: $\begin{cases} 3x - 4y = 0 \\ 2x + 3y = 0 \end{cases} \rightsquigarrow \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ☑

is homogeneous!

Non Ex: $\begin{cases} 3x - 4y = 0 \\ 2x + 3y = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \vec{0}$ ☑

not homogeneous!

<u>Claim:</u> Every homogeneous system has at least 1 solution.

<u>pf:</u> Let $A\vec{x} = \vec{0}$ be a homogeneous linear system.

Setting $\vec{x} = \vec{0}$, $A\vec{0}$ has entry in row i given by $a_{i,1} \cdot 0 + a_{i,2} \cdot 0 + \cdots + a_{i,n} \cdot 0 = 0$, so the ith entry is 0 on left and right. Hence $A\vec{0} = \vec{0}$ is satisfied, and $\vec{x} = \vec{0}$ is a solution to this linear system. $\boxed{//}$.

<u>Prop:</u> Every homogeneous linear system has the zero-solution. (proof above :))

<u>NB:</u> Every linear system has has an associated homogeneous system. (i.e. $A\vec{x} = \vec{b}$ has $A\vec{x} = \vec{0}$).

<u>Claim:</u> The homogeneous system can be used to better understand the original system.

<u>Observation:</u> For A an $m \times k$ matrix and B, C $k \times n$ matrices, we have

$$\star \quad A(B+C) = AB + AC$$

(i.e. matrix multiplication distributes over matrix addition :))

<u>Suggested exercise:</u> show

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}. \quad \star$$

Lem: Suppose $A\vec{x} = \vec{0}$ has solution $\vec{k}$ and $A\vec{x} = \vec{b}$ has solution $\vec{p}$. Then $\vec{p}+\vec{k}$ is a solution to $A\vec{x} = \vec{b}$.

pf: Suppose $A\vec{k} = \vec{0}$ and $A\vec{p} = \vec{b}$.

Then $A(\vec{p}+\vec{k}) = A\vec{p} + A\vec{k} = \vec{b} + \vec{0} = \vec{b}$

Hence $A\vec{x} = \vec{b}$ also has $\vec{x} = \vec{p} + \vec{k}$ as a solution. $\square$

NB: $\vec{k}$ was named for "kernel solution" whereas $\vec{p}$ was named for "particular solution".

Prop: If $\vec{k}$ solves the homogeneous system $A\vec{x} = \vec{0}$ and $\vec{p}$ solves system $A\vec{x} = \vec{b}$, then $\vec{k}+\vec{p}$ solves $A\vec{x} = \vec{b}$